SCATTERING OF ACOUSTIC AND ELASTIC WAVES BY CRACKLIKE OBJECTS: THE ROLE OF HYPERSINGULAR INTEGRALS

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INTRODUCTION

Cracks in NDE are often modeled as slender - shaped voids with little enclosed volume, or as cracks with asperities, or thin slits cut into the surface of a solid, or as touching surfaces with zero enclosed volume, i.e. a 'mathematical' crack (see Fig. 1). Any of these models, among others, may be employed depending on a variety of factors. Although the models with nonzero enclosed volume usually represent reality better, the 'mathematical' model is used very often because of its simplicity and utility, despite certain analytical and numerical difficulties associated with the zero volume aspect. Nevertheless, the more realistic model presents difficulties because of the thinness of the shapes enclosed by the crack surfaces. When such models are used in computations, difficulties with at least the following two features arise: (i) poor conditioning of the final system of equations and (ii) numerical inaccuracy. Both features are due to the proximity of the crack surfaces to each other. This paper demonstrates how a combination of conventional and hypersingular boundary integral equations provides a formulation for scattering of waves from thin - body shapes which is free of the difficulties (i) and (ii). The methodology should be valuable in solving the rough crack and partially - closed crack, as well as the incompletely bonded crack or thin - body inclusion problem. Numerical results are given in this paper for scattering of acoustic waves from certain thin cracklike shapes and data are compared in the near and far field with data from a mathematical crack model. The vector counterpart of such problems, i.e. scattering of elastic waves from cracklike objects, is part of our ongoing research and will be discussed in a future paper.

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FORMULATION

If a time - harmonic acoustic wave impinges upon a thin screen as shown in Fig. 2, the total acoustic field u is given by the representation integral (cf. [1])

$$u(\boldsymbol{\xi}) = \int_{S_T} \left[G(\boldsymbol{x}, \boldsymbol{\xi}) q(\boldsymbol{x}) - \frac{\partial G(\boldsymbol{x}, \boldsymbol{\xi})}{\partial n} u(\boldsymbol{x}) \right] dS(\boldsymbol{x}) + u^I(\boldsymbol{\xi}) \tag{1}$$

where $u \equiv u_s + u^I$, with the s subscript and I superscript indicating 'scattered' and 'incident' fields, respectively; $q = \partial u / \partial n$ on the scatterer surface S_T, G is the free - space Greens function $\exp(ikr)/r$, where $k = \omega/c$, ω is the wave frequency and c is the acoustic wave speed, and dependence of all quantities on ω is understood. Point $\boldsymbol{x}$ is always on S_T, point $\boldsymbol{\xi}$ is off of S_T except when it has a o subscript, i.e. $\boldsymbol{\xi}_o$, in which case it too is on S_T. The singularities or near singularities in G and its derivatives as $\boldsymbol{x} \to \boldsymbol{\xi}_o$ are a key item of concern in this paper.

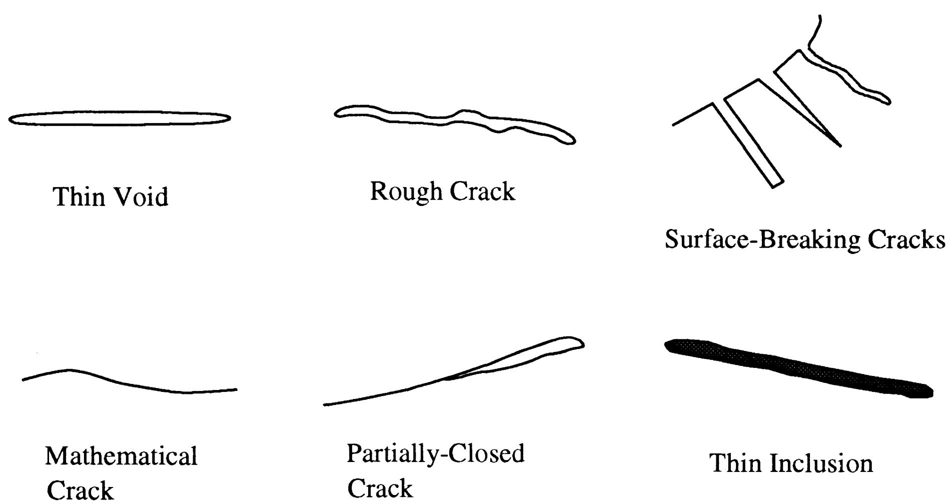

Fig. 1 Cracklike scatterers

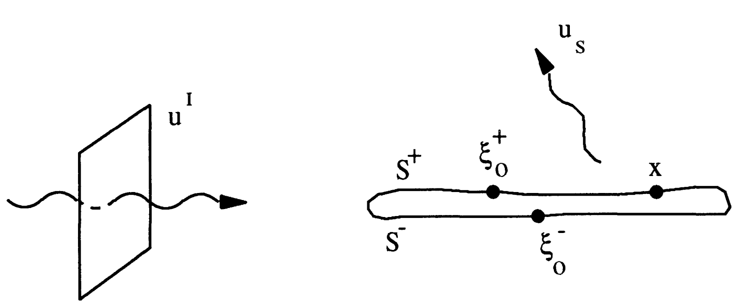

Fig. 2 Wave scattering from a thin screen S_T

Consider for definiteness, and without loss in generality for the issues relevant to this paper, a thin rigid (cracklike) scatterer S_T on the surface of which $q = 0$. Next, if two surfaces S^+ and S^- of S^T are identified between which there is a small but nonzero volume (Fig. 2), the limit in Eq. (1) as $\boldsymbol{\xi} \to \boldsymbol{\xi}_o$ results in

$$\frac{1}{2}u^+(\boldsymbol{\xi}_o)+\int\!\!\!\!\!-_{S^+} \frac{\partial G(\boldsymbol{x}^+,\boldsymbol{\xi}_o{}^+)}{\partial n^+(\boldsymbol{x}^+)}u^+(\boldsymbol{x}^+)\,dS(\boldsymbol{x}^+)+\int_{S^-} \frac{\partial G(\boldsymbol{x}^-,\boldsymbol{\xi}_o{}^+)}{\partial n^-(\boldsymbol{x}^-)}u^-(\boldsymbol{x}^-)\,dS(\boldsymbol{x}^-) = u^I(\boldsymbol{\xi}_o{}^+). \tag{2}$$

Another equation may be written, identical in form, if the limit in Eq. (1) is taken as $\boldsymbol{\xi} \to \boldsymbol{\xi}_o{}^-$ rather than $\boldsymbol{\xi}_o{}^+$ as before. These equations, in particular, may be solved for the unknown quantities $u^+(\boldsymbol{x}^+)$ and $u^-(\boldsymbol{x}^-)$ on S^+ and S^-, respectively, once u^I is specified on each surface. Then the scattered field u for any $\boldsymbol{\xi}$ is given by Eq. (1) to complete the solution to the scattering problem.

Unfortunately, when S_T is thin, i.e. S^+ and S^- very close to each other, integral equation (2) and its counterpart with $\boldsymbol{\xi}_o{}^+$ replaced by $\boldsymbol{\xi}_o{}^-$, have poor properties with respect to yielding a solution for u on S^+ and S^-. Indeed, this pair of equations degenerates into one as the volume between S^+ and S^- vanishes, and this degeneracy is well known in the boundary integral literature on scattering from cracks (e.g. [2]).

What is usually done to overcome this degeneracy is as follows. First, take the limit as S^+ and S^- become indistinguishable and note that $G(\boldsymbol{x}, \boldsymbol{\xi})$ has identical values for $\boldsymbol{x}$ on S^+ and S^- and $\partial G(\boldsymbol{x}^+, \boldsymbol{\xi})/\partial n(\boldsymbol{x}^+) = -\partial G(\boldsymbol{x}^-, \boldsymbol{\xi})/\partial n(\boldsymbol{x}^-)$, such that Eq. (2) takes the form

$$\frac{1}{2}\Sigma u(\boldsymbol{\xi}_o) + \int\!\!\!\!\!-_S \frac{\partial G(\boldsymbol{x}, \boldsymbol{\xi}_o)}{\partial n(\boldsymbol{x})}\Delta u(\boldsymbol{x})\,dS(\boldsymbol{x}) = u^I(\boldsymbol{\xi}_o) \tag{3}$$

in which S is either S^+ or S^- and there is no distinction between $\boldsymbol{\xi}_o{}^+$ and $\boldsymbol{\xi}_o{}^-$, and where $\Delta u \equiv u^+ - u^-$ and $\Sigma u \equiv u^+ + u^-$ are both unknown across S. Now even though there is only one surface S to worry about, and Eq. (3) therefore has none of the poor properties of Eq. (2) plus its $\boldsymbol{\xi}_o{}^-$ counterpart, Eq. (3) by itself is insufficient to determine either Σu or Δu. Thus, the next step is usually to take the normal gradient of Eq. (1) at $\boldsymbol{\xi}_o$, introduce Δu again in the limit as S^+ and S^- coincide to obtain

$$\int\!\!\!\!\!=_S \frac{\partial^2 G(\boldsymbol{x}, \boldsymbol{\xi}_o)}{\partial n(\boldsymbol{x})\partial n(\boldsymbol{\xi}_o)}\Delta u(\boldsymbol{x})\,dS(\boldsymbol{x}) = q^I(\boldsymbol{\xi}_o). \tag{4}$$

Since q^I is known from u^I, Δu on S is the only unknown in Eq. (4). It may be solved from Eq. (4) alone such that Eq. (3) yields Σu, if desired, and the field off of S is given by the appropriate version of Eq. (1) wherein S_T is replaced by S and $u(\boldsymbol{x})$ is replaced by $\Delta u(\boldsymbol{x})$.

Several things should be noted about the strategy above. First, the scatterer in the nondegenerate formulas (3) and (4) is modelled with one surface S, thus it is arbitrarily thin (or a crack); reference to the separate surfaces S^+ and S^- and the actual thickness of the scatterer is lost. Second, the 'dash' through the integral in Eq. (3) and the 'double dash' in Eq. (4) signify special interpretation of strongly singular and hypersingular integrals, respectively, which in turn requires special care in numerical implementation (see [3], [4], [5]). Despite these factors, Eq. (3) and Eq. (4) and their vector counterparts for cracks have been used with much success and are the vehicle for providing important scattering data for a variety of physical problems especially in NDE (e.g. [6], [7], [8]).

Nevertheless, for situations where the thickness (however small) of the scatterer may be important, e.g. in the near field for rough cracks, nonconstant - thickness scatterers such as airfoils, tapered shells, thin inclusions, etc., an alternative is presented here to the mathematically - thin model and its integral formulation as described above. Specifically, if the normal gradient of Eq. (1) is taken and the limit of this gradient expression evaluated on the thin scatterer as $\boldsymbol{\xi} \rightarrow \boldsymbol{\xi}_o{}^-$ (rather than $\boldsymbol{\xi}_o{}^+$), there results (for $q = 0$)

$$\int_{S^-} \frac{\partial^2 G(\boldsymbol{x}^+, \boldsymbol{\xi}_o{}^-)}{\partial n^+(\boldsymbol{x}^+)\partial n^-(\boldsymbol{\xi}_o{}^-)} u^+(\boldsymbol{x}^+)\, dS(\boldsymbol{x}^+) + \neq\!\!\!\!\!\!\int_{S^-} \frac{\partial^2 G(\boldsymbol{x}^-, \boldsymbol{\xi}_o{}^-)}{\partial n^-(\boldsymbol{x}^-)\partial n^-(\boldsymbol{\xi}_o{}^-)} u^-(\boldsymbol{x}^-)\, dS(\boldsymbol{x}^-) = q^I(\boldsymbol{\xi}_o{}^-). \tag{5}$$

A formulation for the thin body problem which uses Eq. (2) and Eq. (5) permits us to separately identify each surface S^+ and S^-, and this formulation is nondegenerate no matter how close to each other S^+ and S^- may be. Thus, by collocating with Eq. (2) on S^+, but integrating over both S^+ and S^-, and by collocating with Eq. (5) on S^-, and integrating over both S^+ and S^- as well, scattering from arbitrarily thin scatterers may be treated without necessity of recourse to the crack or arbitrarily - thin, single - surface model. Strong and hypersingular integrals still need to be dealt with but, as mentioned, this is becoming routine (cf. [5]). It is especially important to note that the fictitious eigenfrequency difficulty (cf. [9]) associated with integral formulations for finite volume (noncrack) scatterers is not present in the formulation Eq. (2) plus Eq. (5).

Additional insight into the nondegenerate character of this formulation can be gained by examining, <u>in discretized form</u>, Eq. (2), Eq. (5) and the combination of Eq. (2) and Eq. (5), as suggested herein for the thin body scatterer. A thin body whose two surfaces are almost flat is assumed for convenience. If elements are introduced on S^+ with say the mirror image of those elements on S^- for a uniformly thin scatterer, the matrix form of Eq. (2) and its counterpart with $\boldsymbol{\xi}_o{}^-$ replacing $\boldsymbol{\xi}_o{}^+$ can be written

$$\begin{bmatrix} \frac{1}{2}I & \sim \frac{1}{2}I \\ \sim \frac{1}{2}I & \frac{1}{2}I \end{bmatrix} \begin{Bmatrix} u^+ \\ u^- \end{Bmatrix} = \begin{Bmatrix} {}^I u^+ \\ {}^I u^- \end{Bmatrix} \tag{6}$$

where the columns represent discrete nodal values of u and ${}^I u$ on the $+$ and $-$ surfaces, and the terms $\sim \frac{1}{2}I$ in the square matrix become $\frac{1}{2}I$ in the limit as S^+ and S^- coincide. Thus the near degeneracy spoken of above is apparent in Eq. (6) for thin scatterers. I is a unit matrix whose size is $n \times n$ when there are n nodes on S^+ and n nodes on S^-. Even when the thin body is not flat the resulting system of equations will be illconditioned. A similar version for the discretized version of Eq. (5) takes the form

$$\begin{bmatrix} a & \sim (-a) \\ \sim (-a) & a \end{bmatrix} \begin{Bmatrix} u^+ \\ u^- \end{Bmatrix} = \begin{Bmatrix} {}^I q^+ \\ {}^I q^- \end{Bmatrix} \tag{7}$$

wherein a (a square matrix of size $n \times n$) is a collection of nonzero coefficients, the specific values of which come from integrals like those in Eq. (5). Again the near degeneracy is apparent.

However, if Eq. (2) and Eq. (5) are used with collocation points $\boldsymbol{\xi}_o{}^+$ and $\boldsymbol{\xi}_o{}^-$, respectively, as indicated in the formulas, but not with both collocation points for each formula as used in Eq. (6) and Eq. (7), there results

$$\begin{bmatrix} \frac{1}{2}I & \sim \frac{1}{2}I \\ \sim(-a) & a \end{bmatrix} \begin{Bmatrix} u^+ \\ u^- \end{Bmatrix} = \begin{Bmatrix} {}^Iu^+ \\ {}^Iq^- \end{Bmatrix}. \tag{8}$$

The matrix of coefficients in Eq. (8) has none of the obviously bad properties of those in Eq. (6) and Eq. (7) and, indeed, Eq. (8) is found to be well conditioned for any scatterer thickness.

There remains the concerns, however, of the ability to compute the entries in the square matrix Eq. (8) with sufficient accuracy in light of the proximity of neighboring elements on S^+ and S^- across a small scatterer thickness. Such computational difficulties for slender bodies of all types have been formidable in boundary element analysis for a long time, and this would be sufficient to negate the nice properties of Eq. (2) plus Eq. (5) (hence Eq. (8)) if the required accuracy could not be achieved. Indeed, this accuracy question alone would favor the single - surface rather than the separate S^+ and S^- surface model of the thin scatterer. However, we demonstrate below that required accuracy can be achieved in integrating over nearby elements on S^+ and S^- such that Eq. (8) may be regarded as a sound and workable formula. Details on how this is done can be found in [10], but it suffices here to note that the essential idea in removing the accuracy difficulty involves the two - term Taylor series expansion (cf. [5]) for the density function in the singular and near singular integrals which reduces all of the near - singular and singular integrals with strong and hypersingular kernels to weakly - singular form.

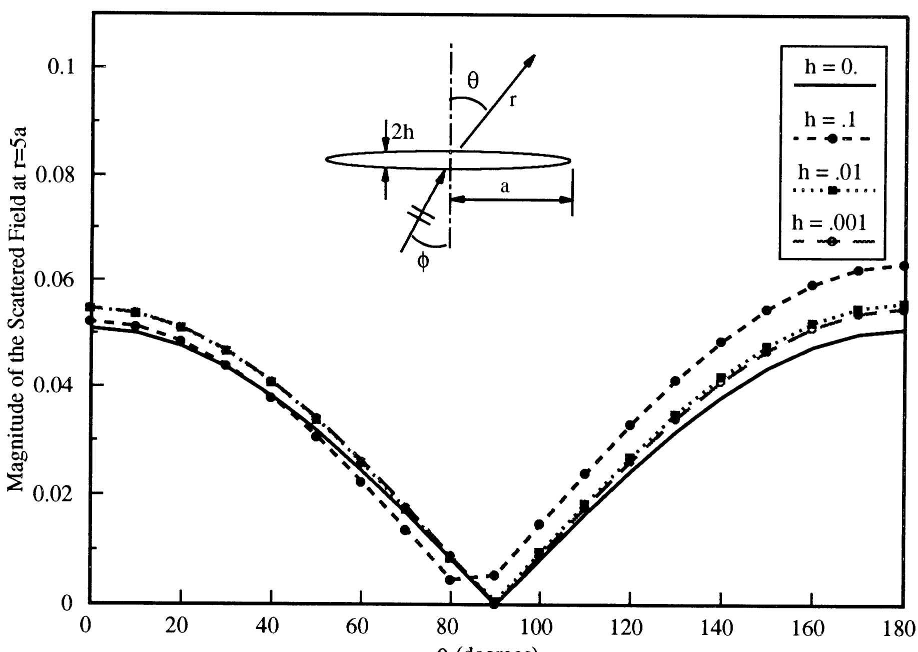

Fig.3 Scattered Field, Normal Incidence φ =0, ka=1, 10 elements, 80 nodes.

NUMERICAL EXAMPLE

Consider a thin rigid scatterer whose shape is obtained by taking a sphere of unit radius and multiplying the x_3 component of all points by h, where $h < 1$. By varying h, scatterers of different thicknesses can be obtained, including the zero - thickness scatterer (crack) where $h = 0$. However, as mentioned earlier, the scattering from a crack is based on Eq. (4) and involves only one of the two surfaces while the $h \neq 0$ model is based on Eq. (8). In this section the scattering of acoustic waves from the $h \neq 0$ model is compared with the crack model. The integral equations are solved using the boundary element method, where each of the sides of the scatterer is discretized by conforming elements to describe the surface and nonconforming elements to describe the boundary variables [5]. For the $h = 0$ model, the square root behavior of the solution along the element edges is built into the elements.

In Fig. 3 the scattered field at $r = 5a$ for different scatterer thicknesses is compared with the $h = 0$ case at a $ka = 1$ and normal incidence. In the far field, it appears that the scattered field for $h \leq 0.1$ is reasonably close to the $h = 0$ case. The difference in the scattered field for the $h = 0.001$ (almost a crack) and $h = 0$ case is mostly due to the presence of the square root behavior of the solution for the $h = 0$ case. This difference for the $h = 0.001$ and $h = 0$ model was observed to reduce with increasing ka (not shown). The specular scattering at different radial distances and $ka = 5$ from a scatterer with $h = 0.1$ are shown in Fig. 4. The comparison with the crack model shows that the difference is less in the far field than in the near field. If the scatterer, instead of being flat, had some structure to it as shown in Fig. 5, a ripple model, then the backscatter differs from the crack model especially when the incident wave is along the plane of the ripple scatterer.

DISCUSSION

The scattering of elastic waves from cracks, thin voids, surface - breaking cracks and thin inclusions is important for NDE. In such problems the scatterer is usually modeled as a mathematical crack. For most practical situations, especially when the far field results are of interest, the crack model is a reasonable one. However, there are many cases where the crack model is insufficient. In this paper the mathematical difficulties, such as degeneracy of the integral equations and the near singular integrals, which arise when the scatterer is thin are discussed. These difficulties are due to the closeness of the two surfaces of the scatterer and they are absent for a crack. Here we suggest a formulation involving the integral equation and its normal gradient which is free from the degeneracy difficulty.

Here we also discuss the degeneracy of the integral equations by looking at the discretized form. It is very clear from the discretized form how the boundary integral equation and its normal gradient when used independently are degenerate and how when used together, as suggested, are no longer degenerate. Such an understanding is not straight forward from the integral equations alone.

The near - singular integration used in this work is an extension of the regularization procedure used for singular integrals in our earlier work and will be discussed elsewhere. The discretization of the two surfaces of the scatterer is a function of the wave number only and not a function of the closeness of the surfaces.

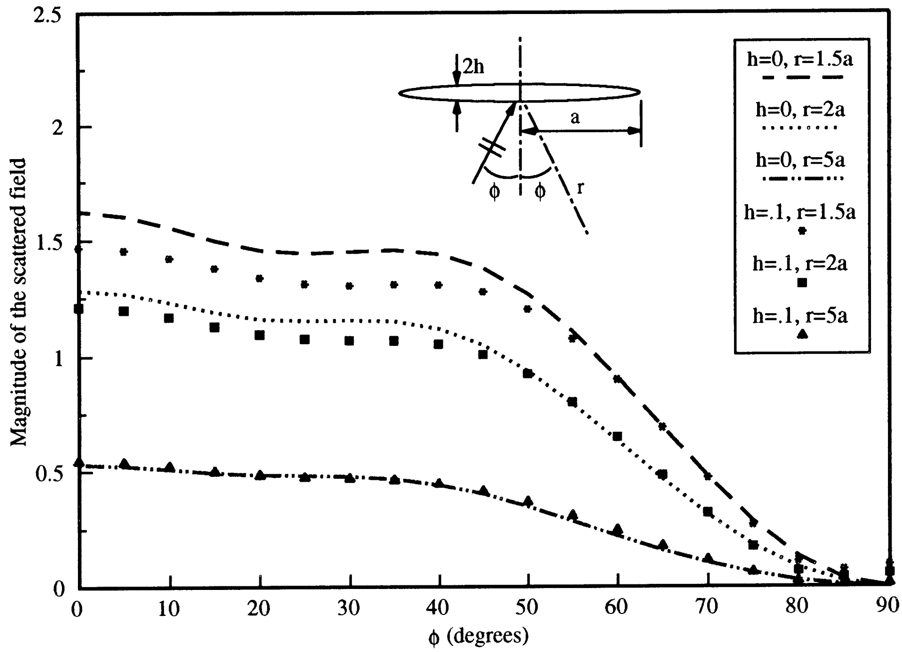

Fig.4 Specular Scattering, h=.1, ka=5, 40 elements, 320 nodes.

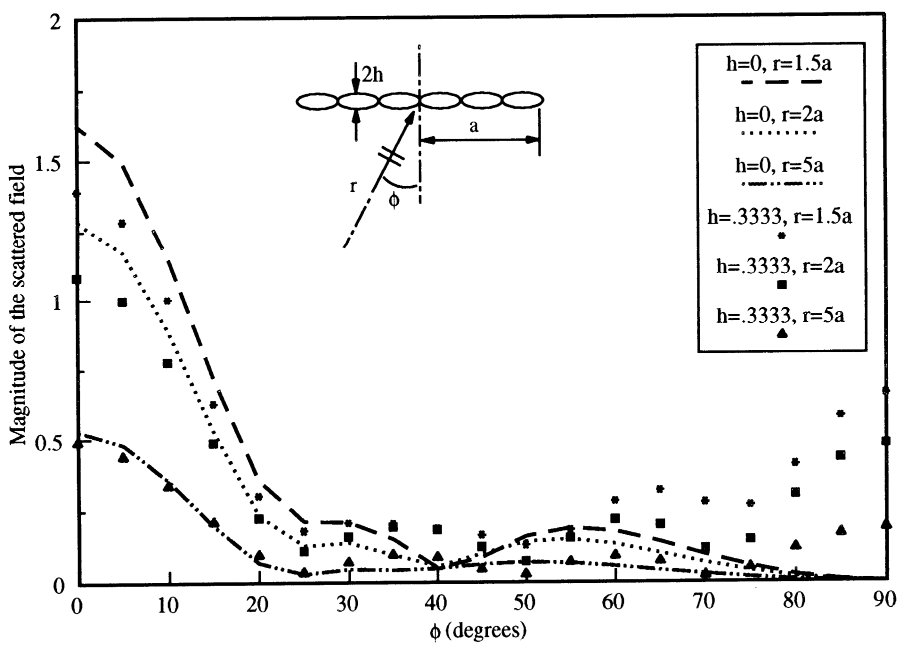

Fig.5 Backscattering, h=.3333, ka=5, 40 elements, 320 nodes.

The scattered field from scatterers of different thicknesses are compared with a zero - thickness (crack) model. Such a study shows that a crack model is a fairly good model for scatterers thinner than a tenth of its length. However this is not always true, particularly when the scatterer surface is not flat but has some structure to it.

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